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Dynamics on differential one-forms

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Mathematical models of dynamics employing exterior calculus are shown to be mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of characteristic differential equations and a characteristic tangent vector which define transformations of the system. This principle, whose origin is Arnold's use of exterior calculus to describe Hamiltonian mechanics and geometric optics, is applied to irreversible thermodynamics and the dynamics of black holes, electromagnetism and strings. It is shown that "exterior calculus" models apply to systems for which the direction of change is given by a characteristic tangent vector and "conventional calculus" models apply to systems whose direction of change is arbitrary. The relationship between the two types of models is shown to imply a technical definition of equilibrium for a dynamic system.

KEY WORDS: applied differential geometry, exterior calculus

1. Introduction

In dynamics, mathematical models that employ calculus use both the exterior calculus of Cartan and the conventional calculus for quantitative description of natural phenomena. So adequate is the agreement between the conclusions of these models and the observables in the experimental domain that Hamiltonian mechanics, irreversible thermodynamics, geometric optics and classical field theories are technically understood only with the geometric objects (e.g., differential forms) of exterior calculus. In contrast (in reference to calculus), quantum mechanics, reversible thermodynamics, physical optics and quantum field theories are not understood in this way, but rather with the geometric objects (e.g., exact differentials) of conventional calculus. A survey of the literature reveals a current period for applications of exterior calculus to dynamics that is comparable to the beginning of the eighteenth century for applications of conventional calculus.

It is shown that the above mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of characteristic differential equations and a characteristic tangent vector which define transformations of the system. This principle, whose origin is Arnold's [1] use of exterior calculus to describe Hamiltonian mechanics and geometric optics, is applied to irreversible thermodynamics and the dynamics of black holes, electromagnetism and strings. It is shown that "exterior calculus" models apply to systems for which the direction of change is given by a characteristic tangent vector and "conventional calculus" models apply to systems whose direction of change is arbitrary. The relationship between the two types of models is shown to imply a technical definition of equilibrium for a dynamic system.

2. Dynamics on differential one-forms

2.1. Differential one-forms

Let us begin by recalling some information about exterior calculus [1,2]. The exterior derivative of a scalar function f (a differential one-form $\mathbf{d}f$) has the same effect on f as the exact differential $\mathrm{d}f$ in conventional calculus; namely, it represents an infinitesimal change in a function f induced by an arbitrary displacement of a point. However, $\mathrm{d}f$ is already a scalar, whereas $\mathbf{d}f$ must be contracted with a tangent vector $\boldsymbol{v}$ to become a scalar. The operation of contraction, denoted by $\mathbf{d}f(\boldsymbol{v})$, thus removes the arbitrariness in the direction of the displacement, where this direction is the same as that of the tangent vector $\boldsymbol{v}$ (tangent vectors and the exterior derivative operator are denoted by italicized boldface symbols and a boldface $\mathbf{d}$, respectively). In this setting, consider an n-dimensional differentiable manifold M with n local coordinates x^k. At every point of M,

(1) there exists a basis set of tangent vectors $\{\partial/\partial\mathbf{x}^k\}$ for an n-dimensional vector space of tangent vectors $\boldsymbol{v}$ belonging to tangent space TM_x and

(2) there exists a basis set of one-forms $\{\mathbf{d}x^k\}$ for an n-dimensional vector space of one-forms $\mathbf{d}f$ on tangent space TM_x.

The tangent bundle $TM(\bigcup_x TM_x)$ and cotangent bundle $T^*M(\bigcup_x T^*M_x$, where T^*M_x – dual of $TM_x)$ have the natural structure of differentiable manifolds of dimension $2n$, with local coordinates $(x^k, \mathbf{d}x^k(\boldsymbol{v}))$ and $(x^k, \mathbf{d}f(\partial/\partial\mathbf{x}^k))$, respectively. A differential one-form $\mathbf{d}S$ on T^*M_x is defined by the contractions $\mathbf{d}S(\xi) = \mathbf{d}f(\boldsymbol{v})$, where $\xi \in T(T^*M_{x)}$; hence,

$$\mathbf{d}S = \mathbf{d}f\big(\partial/\partial\mathbf{x}^k\big)\mathbf{d}x^k. \tag{1}$$

2.2. Dynamics

In Arnold's treatment of Hamiltonian mechanics [1] and in the present examples of dynamic systems, a temporal coordinate x^0 is introduced as an additional local coordinate for M, TM and T^*M, thereby changing TM and T^*M into odd-dimensional manifolds. As a result, an additional term $b_0\mathbf{d}x^0$ is added to equation (1), where b_0 is defined to be a function of all $(2n+1)$ coordinates; hence, b_0 describes the phase flow in this "extended" cotangent space and is called the characteristic function on the extended cotangent bundle. Using b_k for $\mathbf{d}f(\partial/\partial\mathbf{x}^k)$ and $\Omega\mathbf{d}x^0$ for $b_0\mathbf{d}x^0$, we now have

$$\mathbf{d}S = b_k\mathbf{d}x^k + \Omega\big(x^0, \dots, x^n, b_1, \dots, b_n\big)\mathbf{d}x^0. \tag{2}$$

In Hamiltonian mechanics b_k, Ω and x^0 are represented by the momenta, Hamiltonian and time, respectively, but for the examples discussed in section 3, other variables will play the role of b_k, Ω and x^0, as well as of S and x^k. Hence, for the remainder of this section we present the geometry of extended phase space in a general setting that not only applies to geometrical optics and Hamiltonian mechanics (which defines this geometry) but also to irreversible thermodynamics and the dynamics of strings, black holes and electromagnetism.

Arnold's procedure begins by taking the exterior derivative of $\mathbf{d}S$ to get the following differential two-form:

$$\mathbf{d}\omega = \mathbf{d}b_k \wedge \mathbf{d}x^k + \big[\big(\partial\Omega/\partial x^k\big)\mathbf{d}x^k + (\partial\Omega/\partial b_k)\mathbf{d}b_k\big] \wedge \mathbf{d}x^0, \tag{3}$$

where $\omega \equiv \mathbf{d}S$. If x^k and b_k are to describe mappings of the temporal coordinate onto the direction of the system phase flow, then x^k and b_k must be functions of x^0 alone, and vector $\boldsymbol{\xi}$, where

$$\boldsymbol{\xi} = \big(\mathrm{d}b_k/\mathrm{d}x^0\big)\partial/\partial\mathbf{b}_k + \big(\mathrm{d}x^k/\mathrm{d}x^0\big)\partial/\partial\mathbf{x}^k + \partial/\partial\mathbf{x}^0 \tag{4}$$

must satisfy at each point (b_k, x^k, x^0) of the transformation, the equation

$$\mathbf{d}\omega(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \tag{5}$$

for arbitrary tangent vector $\boldsymbol{\eta}$ at each point. This contraction of differential 2-form $\mathbf{d}\omega$ is a mapping of a pair of vectors into an oriented surface, a mapping defined only if the coordinates $\mathrm{d}x^k/\mathrm{d}x^0$ and $\mathrm{d}b_k/\mathrm{d}x^0$ of $\boldsymbol{\xi}$ have the values

$$\big(\mathrm{d}x^k/\mathrm{d}x^0\big) = -(\partial\Omega/\partial b_k) \quad \text{and} \quad \big(\mathrm{d}b_k/\mathrm{d}x^0\big) = \big(\partial\Omega/\partial x^k\big). \tag{6}$$

These equations define the relationship between coordinates $(\mathrm{d}b_k/\mathrm{d}x^0, \mathrm{d}x^k/\mathrm{d}x^0, 1)$ and coordinate values $(\partial\Omega/\partial x^k, -\partial\Omega/\partial b_k, 1)$ for tangent vector $\boldsymbol{\xi}$ at each point of the transformation; thus, the arbitrariness in the coordinates of equation (4) is removed. The characteristic tangent vector obtained by replacing the coordinates in equation (4) with the coordinate values in equations (6), is called the vortex vector. This vector gives the direction of the phase flow (the vortex direction), with the vortex lines (integral curves of equations (6)) passing through points of a closed curve called the vortex tube. But note that equations (6) can be derived in other ways, e.g., with the variation principle or with the use of a symplectic manifold to describe Hamiltonian phase flows; however, the vortex vector associated with these equations arises only through derivation with the odd-dimensional approach in exterior calculus. Support for this approach is given by the fact that it is implied by the multi-dimensional Stokes lemma.

Before concluding this section, three points should be made. First, contraction of $\mathbf{d}S$ with the vortex vector, now called $\boldsymbol{R}$, gives

$$\mathbf{d}S(\boldsymbol{R}) = -b_k(\partial\Omega/\partial b_k) + \Omega, \tag{7}$$

where $\mathbf{d}S(\boldsymbol{R})$ is the Lagrangian on extended tangent space $(x^k, \mathrm{d}x^k/\mathrm{d}x^0, x^0)$. Secondly, note that for equation (5) (where the exterior derivative of a characteristic differential one-form is contracted on a pair of tangent vectors and set equal to the unique scalar zero), the analysis refers to vortex tubes which do not end. For vortex tubes which end in an elementary volume, $\mathbf{d}S(\boldsymbol{\xi}, \boldsymbol{\eta})$ is set equal to a unique scalar other than zero. The example of the source dependent Maxwell equations will illustrate the difference in procedure required for such vortex tubes. Lastly, it is noted that transformations of some models for dynamic systems are conventionally represented by a path which is the projection of the system phase flow along the temporal coordinate axis, rather than the path defined by the vortex vector. This representation has led to the belief that the direction of the process for Hamiltonian mechanics and geometric optics is the direction of the time axis, and that time reversal invariance of the characteristic differential equations implies a reversible model for these systems. However, analysis with exterior calculus shows the direction of the phase flow of Hamiltonian dynamic systems is the direction of the vortex vector, a geometric object independent of representation along any axis. Thus, since the time reversal operation cannot reverse this vector if it is not projected on the temporal coordinate axis, the phase flow for such dynamic systems is irreversible [3]. This conclusion leads to the following proposal for all physical processes assumed to proceed in a characteristic direction. Mathematical models of dynamics employing exterior calculus are mathematical representations of the same unifying principle; namely, the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of characteristic differential equations and a vortex vector which define transformations of the system.

2.3. Comparison with the variation principle

Before applying the foregoing principle to describe physical systems, it is useful to define the relationship between this principle and the variation principle (Euler–Lagrange, Hamilton). It is well known that once a consistent action functional is proposed, the variation principle can be used to generate mathematical models such as Hamilton's equations, the Lorentz force law, Maxwell's equations, Yang–Mills equation, the Einstein equation of geometrodynamics and the quantum equations of Schroedinger, Dirac and Klein–Gordon. The relationship between these principles begins by noting that the quantity S of equation (2) is the action functional, whose increment $I(\partial\Sigma)$ is

$$I(\partial\Sigma) = \int_{\alpha'} \mathcal{L}\,\mathrm{d}^4x - \int_{\alpha} \mathcal{L}\,\mathrm{d}^4x = \int_{\partial\Sigma} \mathcal{L}\,\mathrm{d}^4x, \tag{8}$$

where $\mathcal{L}$ is the Lagrangian density. Integrals in the first line of equation (8) are evaluated on path α and neighboring path α' between the same two endpoints; the integral in the second line is evaluated on the boundary $\partial\Sigma$ of the surface Σ enclosed by α and α'.

In the exterior calculus, instead of considering paths between two endpoints, we consider the vortex lines forming a section of vortex tube between two distinct closed curves encircling the tube. Instead of considering the increment as an integral along the boundary $\partial\Sigma$ of the surface defined by paths α' and α, we consider the increment as an integral along the boundary $\partial\sigma$ of a section σ of the vortex tube, as given by

$$I(\partial\sigma) = \int_{\partial\sigma} \omega, \tag{9}$$

where ω is the differential one-form $\mathbf{d}S$. The variation δS is defined as the linear part of the increment $I(\partial\Sigma)$; whereas, the exterior derivative $\mathbf{d}\omega$ is defined as the principal bilinear part of the increment $I(\partial\sigma)$. The variation condition $\delta S = 0$ is a representation of $\mathbf{d}\omega(\xi, \eta) = 0$ as given in equation (5).

In order to generate mathematical models of dynamic systems, the variation principle requires variation of the coordinates to be zero at the endpoints and arbitrary in between. Hence, due to this arbitrariness, when the variation principle is used to develop mathematical models of certain dynamic systems, e.g., Hamiltonian dynamics, the analysis does not make it clear that a vortex vector has been defined, and that some predictions of the model depend as much on this vector as on the fundamental equations for the model. Of course the multi-dimensional Stokes' lemma requires the boundary $\delta\Sigma$ to be an infinitesimal square containing the vortex direction, but the variation method does not calculate the vortex vector for this direction. This statement is supported by the fact that although the integrand in the variation integral is a differential one-form, variation techniques employ conventional calculus. In addition, the multi-dimensional Stokes lemma implies the odd-dimensional approach, but the variation method uses an even-dimensional approach since $\mathcal{L}$ is a function of $2n$ coordinates. Hence, although this use of the variation principle requires the existence of extremal paths and predicts equations to define functions of these paths, this principle leaves arbitrary the definition of the vortex vector for the system. Upon interpreting this arbitrariness as implying equally probable directions and hence no preferred direction, then mathematical models generated by the variation principle technically define systems at equilibrium.

3. Applications

The principle described in section 2 will be illustrated in table 1, with applications to black hole mechanics and thermodynamics, and in table 2 with applications to electromagnetic and string field theories. For each category of dynamics, we have listed the characteristic differential one-form, basis vectors in tangent space, characteristic differential equations, vortex vector and the Lagrangian. Although some of this information is well known, by categorizing it in the form of this chart, the full impact of the following principle emerges: the description of a dynamic system with a characteristic differential one-form on an odd-dimensional differentiable manifold leads, by analysis with exterior calculus, to a set of characteristic differential equations and a characteristic tangent vector which define transformations of the system.

Table 1

Application of model system to Hamiltonian mechanics, geometric optics, black hole mechanics and irreversible thermodynamics.

Dynamics on differential one-forms	Characteristic one-form	Basic vectors in tangent space	Characteristic differential equations	Vortex vector	Lagrangian
$\mathbf{d}S$ – one-form for model dynamic system	$\mathbf{d}S = b_k \mathbf{d}x^k + \Omega \mathbf{d}x^0$	$(\partial/\partial \boldsymbol{b}_k, \partial/\partial \boldsymbol{x}^k, \partial/\partial \boldsymbol{x}^0)$	$\mathrm{d}x^k/\mathrm{d}x^0 = -(\partial\Omega/\partial b_k)$ $\mathrm{d}b_k/\mathrm{d}x^0 = (\partial\Omega/\partial x^k)$	$\boldsymbol{R} = (\Omega_x \; -\Omega_b \; 1)$ $\Omega_x = (\partial\Omega/\partial x^k)$ $\Omega_b = (\partial\Omega/\partial b_k)$	$\mathbf{d}S(\boldsymbol{R}) = -b_k\Omega_{bk} + \Omega$
Hamiltonian mechanics: $\mathbf{d}S_H$ – one-form for action	$\mathbf{d}S_H = p_i \mathbf{d}q^i - H\mathbf{d}t$	$(\partial/\partial \boldsymbol{p}_i, \partial/\partial \boldsymbol{q}^i, \partial/\partial \boldsymbol{t})$	$\mathrm{d}q^i/\mathrm{d}t = (\partial H/\partial p_i)_{qi,t}$ $\mathrm{d}p_i/\mathrm{d}t = -(\partial H/\partial q^i)_{pi,t}$	$\boldsymbol{R} = (-H_{qi} \; H_{pi} \; 1)$	$\mathbf{d}S_H(\boldsymbol{R}) = p_i H_{pi} - H$
Geometric optics: $\mathbf{d}\phi$ – one-form for optical path length	$\mathbf{d}\phi = k_i \mathbf{d}q^i - \omega \mathbf{d}t$	$(\partial/\partial \boldsymbol{k}_i, \partial/\partial \boldsymbol{q}^i, \partial/\partial \boldsymbol{t})$	$\mathrm{d}q^i/\mathrm{d}t = (\partial\omega/\partial k_i)_{qi,t}$ $\mathrm{d}k_i/\mathrm{d}t = -(\partial\omega/\partial q^i)_{ki,t}$	$\boldsymbol{R} = (-\omega_{qi} \; \omega_{ki} \; 1)$	$\mathbf{d}\phi(\boldsymbol{R}) = k_i\omega_{ki} - \omega$
Black hole mechanics: $\mathbf{d}M$ – one-form	$\mathbf{d}M = \Omega\mathbf{d}J + T\mathbf{d}S$	$(\partial/\partial \boldsymbol{\Omega}, \partial/\partial \boldsymbol{J}, \partial/\partial \boldsymbol{S})$	$\mathrm{d}J/\mathrm{d}S = -(\partial T/\partial\Omega)$ $\mathrm{d}\Omega/\mathrm{d}S = (\partial T/\partial J)$	$\boldsymbol{R} = (T_J \; -T_\Omega \; 1)$	$\mathbf{d}M(\boldsymbol{R}) = -\Omega T_\Omega + T$
for mass of black hole	$\mathbf{d}M = \Omega\mathbf{d}J + \kappa'\mathbf{d}A$ $\kappa' = (\kappa/8\pi)$	$(\partial/\partial \boldsymbol{\Omega}, \partial/\partial \boldsymbol{J}, \partial/\partial \boldsymbol{A})$	$\mathrm{d}J/\mathrm{d}A = -(\partial\kappa'/\partial\Omega)$ $\mathrm{d}\Omega/\mathrm{d}A = (\partial\kappa'/\partial J)$	$\boldsymbol{R} = (\kappa'_J - \kappa'_\Omega \; 1)$	$\mathbf{d}M(\boldsymbol{R}) = -\Omega\kappa'_\Omega + \kappa'$
$\mathbf{d}U$ – one-form for internal energy	$\mathbf{d}U = -P\mathbf{d}V + T\mathbf{d}S$	$(\partial/\partial \boldsymbol{P}, \partial/\partial \boldsymbol{V}, \partial/\partial \boldsymbol{S})$	$\mathrm{d}V/\mathrm{d}S = (\partial T/\partial P)_{V,S}$ $\mathrm{d}P/\mathrm{d}S = -(\partial T/\partial V)_{P,S}$	$\boldsymbol{R} = (-T_V \; T_P \; 1)$	$\mathbf{d}U(\boldsymbol{R}) = -PT_P + T$
$\mathbf{d}A$ – one-form for Helmholtz energy	$\mathbf{d}A = -P\mathbf{d}V - S\mathbf{d}T$	$(\partial/\partial \boldsymbol{P}, \partial/\partial \boldsymbol{V}, \partial/\partial \boldsymbol{T})$	$\mathrm{d}V/\mathrm{d}T = -(\partial S/\partial P)_{V,T}$ $\mathrm{d}P/\mathrm{d}T = (\partial S/\partial V)_{P,T}$	$\boldsymbol{R} = (S_V \; -S_P \; 1)$	$\mathbf{d}A(\boldsymbol{R}) = PS_P - S$

Table 2

Application of model system to classical electromagnetism and string mechanics.

Dynamics on differential one-forms	Characteristic one-form	Basic vectors in tangent space	Characteristic differential equations	Vortex vector	Lagrangian
Classical electromagnetism:	$\mathbf{d}S_F = (f_i/e)\mathbf{d}x^i - \Omega\mathbf{d}t$	$(e\partial/\partial \boldsymbol{f}_i, \partial/\partial \boldsymbol{x}^i, \partial/\partial \boldsymbol{t})$	$\mathrm{d}x^i/\mathrm{d}t = e(\partial\Omega/\partial f_i)_{xi,t}$	$\boldsymbol{R} = (-\Omega_{xi}\ e\Omega_{fi}\ 1)$	$\mathbf{d}S_F(\boldsymbol{R}) = f_i\Omega_{fi} - \Omega$
$\mathbf{d}S_F$ – Faraday one -form	$f_i = e[\boldsymbol{E} + (\mathrm{d}\mathbf{x}/\mathrm{d}t) \times \boldsymbol{B}]_i$ $\Omega = (\mathrm{d}\mathbf{x}/dt) \cdot \boldsymbol{E}$		$\mathrm{d}f_i/\mathrm{d}t = -e(\partial\Omega/\partial x^i)_{fi,t}$		
Classical electromagnetism:	$\mathbf{d}S_{*F} = ({}^*f_i/e)\mathbf{d}x^i - \Omega\mathbf{d}t$	$(e\partial/\partial^* \boldsymbol{f}_i, \partial/\partial \boldsymbol{x}^i, \partial/\partial \boldsymbol{t})$	$\mathrm{d}x^i/\mathrm{d}t = e(\partial\Omega/\partial^* f_i)_{xi,t}$	$\boldsymbol{R} = (-\Omega_{xi}\ e\Omega_{*fi}\ 1)$	$\mathbf{d}S_{*F}(\boldsymbol{R}) = {}^*f_i\Omega_{*fi} - \Omega$
$\mathbf{d}S_{*F}$ – Maxwell one-form	${}^*f_i = e[-\boldsymbol{B} + (\mathrm{d}\mathbf{x}/\mathrm{d}t) \times \boldsymbol{E}]_i$ $\Omega = -(\mathrm{d}\mathbf{x}/\mathrm{d}t) \cdot \boldsymbol{B}$		$\mathrm{d}^* f_i/\mathrm{d}t = -e(\partial\Omega/\partial x^i)_{*fi,t}$		
String mechanics:	$\mathbf{d}S_{\mathrm{MN}} = P_i\mathbf{d}x^i + \mathcal{H}\mathbf{d}\tau$	$(\partial/\partial \boldsymbol{P}_i, \partial/\partial \boldsymbol{x}^i, \partial/\partial \boldsymbol{\tau})$	$\mathrm{d}x^i/\mathrm{d}\tau = -(\partial\mathcal{H}/\partial P_i)_{xi,\tau}$	$\boldsymbol{R} = (\mathcal{H}_{xi} - \mathcal{H}_{Pi}\ 1)$	$\mathbf{d}S_{\mathrm{MN}}(\boldsymbol{R}) = -P_i\mathcal{H}_{Pi} + \mathcal{H}$
$\mathbf{d}S_{\mathrm{MN}}$ – Mitra–Nambu one-form	$P_i = \pi_{ij}(\partial x^j/\partial\sigma)$ $\mathcal{H} = [H - p_i(\partial x^i/\partial\tau) - \phi_i(\partial x^i/\partial\sigma)]$		$\mathrm{d}P_i/\mathrm{d}\tau = (\partial\mathcal{H}/\partial x^i)_{Pi,\tau}$		

3.1. Hamiltonian mechanics

Conventionally a symplectic manifold (M^{2n}, ω^2) is used to describe Hamiltonian mechanics, where ω^2 is a closed nondegenerate 2-form. In this case $\mathbf{d}\omega^2(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = 0$ for all $\omega^2(\boldsymbol{X}, \boldsymbol{Y}) \neq 0$ and for all tangent vectors $\boldsymbol{Z}$. The present approach is due to Arnold [1], who introduced an odd-dimensional approach with (M^{2n+1}, ω^1) to develop a mathematical model for Hamiltonian mechanics, where (p_i, q^i) is a conjugate pair, t is the time and $H(q^i, p_i, t)$ is the characteristic function. In Arnold's development it was shown that the multi-dimensional Stokes' lemma directly implies all the basic propositions of Hamiltonian mechanics. Contained in the present development is this approach plus the interpretation of [3], that the vortex vector implies Hamiltonian mechanics is irreversible. This conclusion comes from the proposition that irreversible thermodynamics and Hamiltonian mechanics can be described by the same geometric formalism and that the concept of irreversibility in thermodynamics and therefore in Hamiltonian mechanics, is implied by the vortex vector. The uniqueness of the vortex vector is also seen in the contraction of $\mathbf{d}S_H$ with the vortex vector $\boldsymbol{R}$ to yield the correct Lagrangian $\mathbf{d}S_H(\boldsymbol{R})$, thereby giving an internal check on proposed characteristic differential one-forms for physical systems.

3.2. Geometric optics

For geometric optics, $\mathbf{d}\phi$ is the characteristic differential one-form, ϕ is the optical path length (the "eikonal"), ω is the wave frequency, (k_i, q^i) is the conjugate pair, $\boldsymbol{k}$ is the gradient of the optical path length, $\mathbf{d}q^i$ and $\mathbf{d}t$ are basic differential one-forms for the position and time, and frequency $\omega(q^i, k_i, t)$ is the characteristic function. Noted in this case are the vortex vector and the Lagrangian L, with zero for the Lagrangian in the case of a vacuum. The appearance of the zero has in some cases led to the omission of a variation term for geometric optics comparable to the variation term $\delta \int L\, \mathrm{d}t$ in Hamiltonian mechanics. In agreement with this omission the present results imply that the contraction of differential one-form $\mathbf{d}\phi$ with its vortex vector is zero for the vacuum case only, where $\omega = ck$ and c is the speed of light. The vortex vector is a geometric object giving the direction of change for the optical medium; hence, it is independent of representation on the time axis. This is in sharp contrast to the predictions of the conventional calculus, where the system motion is represented as the result of a projection of the system path along the time axis and is thus subject to time reversal and the resulting interpretations of time reversal invariance. The present results indicate the vortex vector is a key component in the description of geometric optics.

3.3. Black hole mechanics

According to Bardeen, Carter and Hawkins (BCH) the first law of black hole mechanics [4] states that in the vacuum case the relationship between the variation of the mass M, angular momentum J and surface area A of two nearby stationary black holes is given by

$$\delta M = \Omega\delta J + \frac{\kappa}{8\pi}\delta A + \cdots, \tag{10}$$

where Ω is the angular velocity and κ is the surface gravity. BCH formed an analogy of this law with the first law of thermodynamics by considering the relationship between the variation of the energy (proportional to M), work terms (proportional to $\Omega\delta J$), temperature (proportional to κ) and entropy (proportional to A) of two nearby thermal equilibrium states of a single black hole. Hawkins later discovered that the physical temperature T of a black hole is given by $\kappa/(2\pi)$ and, according to the analogy with thermodynamics, the area A is $4S_{\text{bh}}$, where S_{bh} is the physical entropy of a black hole in general relativity. In the present model, we consider the exterior derivatives corresponding to $\text{d}M$, $\text{d}J$, $\text{d}A$ and $\text{d}S$, to obtain

$$\mathbf{d}M = \Omega\mathbf{d}J + \frac{\kappa}{8\pi}\mathbf{d}A = \Omega\mathbf{d}J + T\mathbf{d}S_{\text{bh}}, \tag{11}$$

where we consider only the first two terms of δM in equation (10). Both parts of equation (11) now hold for nonstationary perturbations of a black hole, since differential one-forms are used. Following the model system, κ is the characteristic function $\kappa(\Omega, J, A)$, where Ω and J are the angular velocity and angular momentum, respectively. When using T as the characteristic function $T(\Omega, J, S_{\text{bh}})$, we have the same definitions for Ω and J except that when κ is the characteristic function, Ω and J are mappings of A onto the system phase flow, but when T is the characteristic function, Ω and J are mappings of S_{bh} onto the system phase flow. New proposals in this case are the set of differential equations, vortex vector and Lagrangian for the dynamics of black holes, results that can be given the same physical interpretation as the model system. This analysis can be compared to the following discussion of irreversible thermodynamics.

3.4. Irreversible thermodynamics

Story [3] used differential one-forms to develop a mathematical model for irreversible thermodynamics. The starting point for this development was to identify the differential one-forms $\mathbf{d}U$ and $\mathbf{d}A$ corresponding to exact differentials for the internal energy U and the Helmholtz energy A. $T(V, P, S)$ is the characteristic function for $\mathbf{d}U$ and $S(V, P, T)$ is the characteristic function for $\mathbf{d}A$; V and P are conjugate pairs in each case, where P, V, S and T are, respectively, the system pressure, volume, entropy and temperature. When the technique for applying the proposed principle was utilized, sets of differential equations and a vortex vector for each one-form were obtained. The vortex vectors are interpreted as giving the direction of irreversible change for the system in (P, V, S) and in (P, V, T) space, respectively, where S and T are the corresponding temporal coordinates. The differential equations are the irreversible counterparts of the Maxwell relations in reversible thermodynamics. Contraction of differential one-forms $\mathbf{d}U$ and $\mathbf{d}A$ on their respective vortex vectors gives the Lagrangian for each extended tangent space.

3.5. Classical electromagnetism

For classical electromagnetism, conventionally the exterior calculus formalism [2] leading to Maxwell's equations begins with two characteristic differential two-forms, referred to as the Faraday two-form F and its dual, the Maxwell two-form ${}^*F$, as given by

$$F = B_i \mathbf{d}x^j \wedge \mathbf{d}x^k + E_i \mathbf{d}x^i \wedge \mathbf{d}t \tag{12}$$

and

$${}^*F = E_i \mathbf{d}x^j \wedge \mathbf{d}x^k - B_i \mathbf{d}x^i \wedge \mathbf{d}t, \tag{13}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are magnetic and electric fields, $\mathbf{x}$ is the position and t is the time. Instead of the foregoing even-dimensional manifold and 2-form (M^{2n}, ω^2) for a description of electromagnetism, an odd-dimensional manifold and one-form (M^{2n+1}, ω^1) are introduced. This technique is analogous to Arnold's use of (M^{2n+1}, ω^1) to obtain a mathematical model for Hamiltonian mechanics rather than using a symplectic manifold (M^{2n}, ω^2). In order to use the odd-dimensional approach for electromagnetism, the isomorphism between tangent vector $\boldsymbol{X}$ and differential one-form $\mathbf{d}S_F$ is employed. This isomorphism is established by the contraction $F(\boldsymbol{X})$, where $\mathbf{d}S_F \equiv F(\boldsymbol{X})$ and $\boldsymbol{X} = (\mathrm{d}x^i/\mathrm{d}t)\partial/\partial\mathbf{x}^i + \partial/\partial t$; $\mathbf{d}S_{*F}$ is obtained with the contraction ${}^*F(\boldsymbol{X})$. From this analysis it is noted that the coefficient of $\mathbf{d}x^i$ for $\mathbf{d}S_F$ is the Lorentz force f_i divided by the electric charge e; hence, the gradient of S_F is the Lorentz force divided by the charge. Since $\mathbf{d}S_{*F}$ resulted from the contraction of the dual of F with $\boldsymbol{X}$, the quantity $[-\boldsymbol{B} + (\mathrm{d}\mathbf{x}/\mathrm{d}t) \times \boldsymbol{E}]_i$ is designated as ${}^*f_i/e$. It should be noted that the coordinates of the extended cotangent spaces in this case are $(f_i/e, x^i, t)$ and $({}^*f_i/e, x^i, t)$, with corresponding basis vectors for the respective tangent spaces. Hence the relevant tangent vectors are of the type $\boldsymbol{\xi} = (\mathrm{d}f_i/\mathrm{d}t)\partial/\partial\mathbf{f}_i + (\mathrm{d}x^i/\mathrm{d}t)\partial/\partial\mathbf{x}^i + \partial/\partial t$. Continuing, the procedure of the model dynamic system was then employed for $\mathbf{d}S_F$, thereby generating a set of characteristic differential equations and vortex vector. For $\mathbf{d}S_{*F}$ the vortex tubes end in an elementary charge, rather than nowhere ending as for $\mathbf{d}S_F$; thus, a modification of the procedure of the model dynamic system was required. The unique scalar obtained when following the procedure for this differential one-form is $4\pi\omega^2_{*J}(\boldsymbol{X}, \boldsymbol{Y})$ for arbitrary $\boldsymbol{Y}$, rather than zero, where $\mathbf{d}\omega^2_{*J}$ is the differential three-form ${}^*J$ (dual of charge current J) . Note that differential equations obtained in this way are mathematically equivalent to the Maxwell equations in the following sense: If the inverse of the contraction $F(\boldsymbol{X})$ is performed, F and, through duality, ${}^*F$ are obtained again; then, the exterior derivatives $\mathbf{d}F$ and $\mathbf{d}{}^*F$ can be contracted on a triple of tangent vectors and set equal to the unique scalars zero and $4\pi{}^*J(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$, respectively, thereby generating Maxwell's equations.

In the present analysis characteristic differential one-forms are used as a starting point, thereby leading to vortex vectors, differential equations equivalent to Maxwell's equations and the use of the contractions $\mathbf{d}S_F(\boldsymbol{R})$ and $\mathbf{d}S_{*F}(\boldsymbol{R})$ to obtain expressions for the Lagrangian. Note that the predicted Lagrangian in table 2, obtained by the contraction $\mathbf{d}S_F(\boldsymbol{R})$, is not the Lagrangian density $[(E^2 - B^2)/(8\pi) - \rho\phi + \boldsymbol{J}\cdot\boldsymbol{A}/c]$ used in the variation method to generate Maxwell's equations. However, the present mathematical model is supported by the fact that the mathematical equivalent of Maxwell's equations is generated by the demonstrated procedure, and this procedure correctly predicts the Lagrangian for Hamilton mechanics and geometric optics. The use of these differential equations is straightforward in the sense that the interpretation given to the vortex vector and differential equations of the model system can be given here. For the notation, symbol ρ is the charge density, and $\boldsymbol{A}$ and ϕ are vector and scalar electric potentials.

Maxwell's equations remain unchanged despite developments in quantum mechanics and special and general relativity. The ease of inclusion of Maxwell's equations into the framework of the present development lends further support to the general applicability of the proposed principle.

3.6. String mechanics

Following the work of Nambu, Mitra [5] studied the use of a differential 2-form for the dynamics of weighted strings. In order to describe the sheet traced out in spacetime by a moving string, Mitra introduced two independent variables σ and τ to replace the single variable t for time, and used three independent variables for the momenta, where this 2-form is given by

$$\omega_{\mathrm{M}}^2 = p_i \mathrm{d}\sigma \wedge \mathrm{d}x^i + \phi_i \mathrm{d}x^i \wedge \mathrm{d}\tau + \frac{1}{2}\pi_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j - H\mathrm{d}\sigma \wedge \mathrm{d}\tau, \tag{14}$$

where

$H = H(p_i, \phi_i, \pi_{ij}, x^i)$;

$\sigma, \tau -$ two symbols for time, σ is considered as a parameter labeling different points of the string and τ is the symbol for time as in Hamiltonian mechanics;

$x^i(\sigma, \tau) -$ position coordinate;

$p_i(\sigma, \tau) -$ momentum coordinate;

$\phi_i(\sigma, \tau), \pi_{ij}(\sigma, \tau) -$ two sets of momenta determined in terms of x^i and p_i by constraints; $\pi_{ij} = -\pi_{ji}$ and so only $n(n-1)/2$ of the π_{ij} are independent.

Following the approach taken for electromagnetic fields, ω_{M}^2 was represented as a one-form by contracting ω_{M}^2 with tangent vector $\boldsymbol{X}$, giving $\mathbf{d}S_{\mathrm{MN}} \equiv \omega_M^2(\boldsymbol{X})$, where

$$\boldsymbol{X} = \left(\partial x^i/\partial\sigma\right)\partial/\partial\mathbf{x}^i + (\partial p_i/\partial\sigma)\partial/\partial\mathbf{p}_i + (\partial\phi_i/\partial\sigma)\partial/\partial\boldsymbol{\phi}_i + \frac{1}{2}(\partial\pi_{ij}/\partial\sigma)\partial/\partial\boldsymbol{\pi}_{ij} + \partial/\partial\sigma. \tag{15}$$

In this way, a characteristic differential one-form $\mathbf{d}S_{\mathrm{MN}}$ for string dynamics was obtained, where $\mathcal{H}(P_i, x^i, \tau)$ is the characteristic function and (x^i, P_i) is the conjugate pair. Use of the same procedure as for the model dynamic system resulted in a set of differential equations and a vortex vector which describe irreversible transformations of the system defined by $\mathbf{d}S_{\mathrm{MN}}$. The relevant tangent vector used in this case is $\xi = (\mathrm{d}P_i/\mathrm{d}\tau)\partial/\partial\mathbf{P}_i + (\mathrm{d}x^i/\mathrm{d}\tau)\partial/\partial\mathbf{x}^i + \partial/\partial\tau$. Differential equations generated by this procedure are mathematically equivalent to those of Mitra in the following sense: the inverse of the contraction operation can be performed to obtain equation (14), then $\mathbf{d}\omega^2_{\mathrm{M}}$ can be contracted on a triple of tangent vectors and set to zero, thus implying Mitra's equations.

This discussion is concluded by noting that the characteristic function, $\mathcal{H}(P_{i,} x^i, \tau) = \mathcal{H}(\pi_{ij,} x^i, \sigma, \tau)$, is a function of eleven independent coordinates, as given by six π_{ij}, three x^i, one σ and one τ. In other discussions of string theory, namely, in M-theory, more advanced differential geometric techniques are used to represent a string as a two-dimensional surface moving through an eleven-dimensional Calabi–Yau space. In the present discussion, application of the proposed principle is an approach with characteristic differential one-forms as fundamental; implications of the multi-dimensional Stokes' lemma results in a set of characteristic differential equations, the vortex vector and the Lagrangian for the dynamics of string membranes.

4. Conclusion

It has been shown that the use of exterior calculus in dynamics leads to a principle useful for deriving mathematical models of dynamic systems. Development of this principle relied on the fact that all the basic propositions of the geometry of extended phase space are implied by the multi-dimensional Stokes' lemma, and that Hamiltonian mechanics and irreversible thermodynamics are different mathematical representations of this principle. Details of the principle showed that the characteristic direction of displacements inherent to exterior calculus models implies irreversible changes of dynamic systems, whereas the arbitrary direction of displacements inherent to conventional calculus models implies equilibrium. Once the principle was discussed in these general terms, it was possible to demonstrate applications to a diverse set of phenomena, namely, irreversible thermodynamics, black hole mechanics, and electromagnetic and string field theories. For each of these applications, the characteristic differential one-form, the basic vectors in tangent space, the characteristic differential equations, the vortex vector and the Lagrangian were synthesized.

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